Creating a Straight Line from Infinite Numbers of Perfect Circles – an Exploration of Limits in Geometry

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1 Introduction

Although a straight line connecting one point to another can be created with a ruler, in this paper, one theorem will be proposed to suggest a method to create a straight line from infinitely many circles.

2 Theorem

Suppose two perfect circles are drawn, such that the center of each of the two circles lie on the other circle (Figure 1).

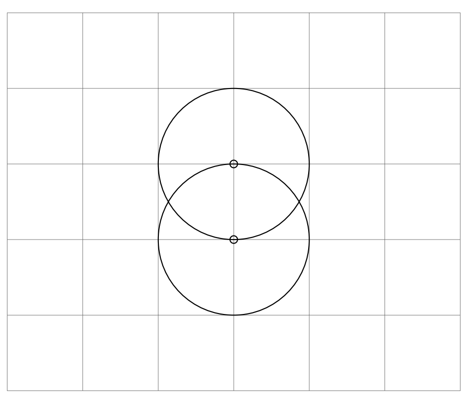

Figure 1: Circles with the centers on the other circle

Suppose another circle is drawn so that the new circle's radius is half that of the first two circles in Figure 1. This new circle will be placed such that the new circle intersects the centers of both of the first two circles (Figure 2).

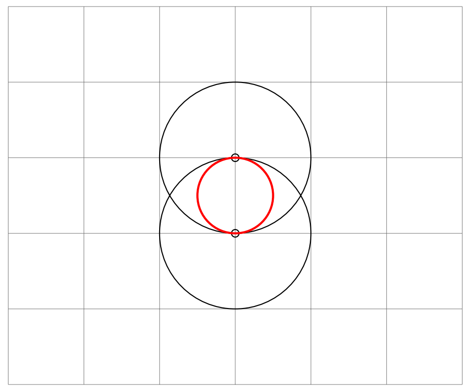

Figure 2: The new circle is drawn (in red), so that it intersects the centers of both of the first two circles.

Another circle is then drawn (Figure 3) such that the center of the red circle in Figure 2 lies on the new circle, and the new circle lies above the first red circle.

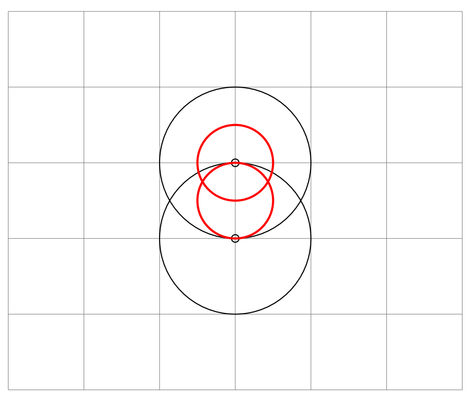

Figure 3: A new circle is drawn such that the first circle's center lies on the new circle.

A new circle is then drawn between the part where the two red circles overlap like in Figure 2, and the whole process is repeated. After repeating the process for two more times, the figure will appear like in Figure 4.
For each layer, the points where the two-circle intersects form a straight line to the center of the upper circles in each layer as shown in Figure 5, where a green

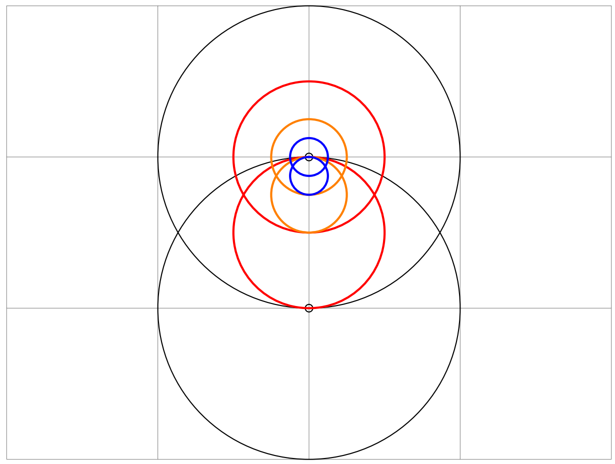

Figure 4: The same process was repeated two more times, resulting in two new layers (in orange and blue, respectively).

line is drawn to visualize this. If the process is repeated for infinite number of times or in other words, if the (number of layers) $\to \infty$, the points that line up for the "layers" closest to the center of the upper circles in each layer will be infinitesimally close to each other that a straight line can be created.

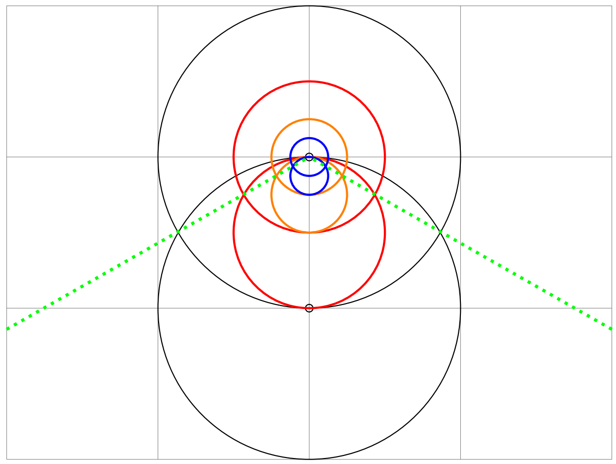

Figure 5: A green dotted line is used to show where the points line up.

3 Proof

The theorem stated can be mathematically proven as follows:

Lemma 1 *An equation of a circle is*

$$(y-\alpha)^2 + (x-\beta)^2 = r^2 \tag{1}$$

where (x, y) are points in cartesian coordinate, r is the radius of the circle, and α and β are shifts in the positive y and x axis, respectively.

Suppose there is a circle A and circle B, both of radius r, such that A and B are drawn like in Figure 1, so that the center of each circle is on another circle. Suppose also that circle A is centered at the origin.

Corollary 1.1 *Circle A's equation will be*

$$y^2 + x^2 = r^2 \tag{2}$$

and Circle B's equation will be

$$(y-r)^2 + x^2 = r^2 \tag{3}$$

With this, the point of intersect between the circles C and D can be calculated as follows:

$$(y-r)^2 = y^2 \tag{4}$$
$$(y^2 - 2ry + r^2) - y^2 = 0 \tag{5}$$
$$r^2 - 2ry = 0 \tag{6}$$
$$y = \frac{1}{2}r \tag{7}$$

The x-axis can be calculated as follows using the equation of of one of the circles in Corollary 1.1:

$$y^2 + x^2 = r^2 \tag{8}$$
$$\left(\frac{1}{2}r\right)^2 + x^2 = r^2 \tag{9}$$
$$x^2 = \frac{3}{4}r^2 \tag{10}$$
$$x = \pm\frac{\sqrt{3}}{2}r \tag{11}$$

Corollary 1.2 *The circle A and Circle B will intersect each other at points* $\left(\frac{\sqrt{3}}{2}r, \frac{1}{2}r\right)$ *and* $\left(-\frac{\sqrt{3}}{2}r, \frac{1}{2}r\right)$.

Suppose we have circle C, such that the centers of circles A and B lie on it (Figure 2). This circle will have half the radius of circles A and B, and therefore will have the radius $\frac{1}{2}r$. The circle will also be centered at a point with the y-coordinate corresponding to the points of intersection between A and B, which is $\frac{1}{2}r$.

Corollary 1.3 *Circle C will be defined according to Lemma 1 as*

$$\left(y - \frac{1}{2}r\right)^2 + x^2 = \left(\frac{1}{2}r\right)^2 \tag{12}$$
$$\left(y - \frac{1}{2}r\right)^2 + x^2 = \frac{1}{4}r^2 \tag{13}$$

Suppose now that a circle D, which is the same size as C is drawn such that, similar to cases in A and B, centers of circles C and D are on the other circle (Figure 3). As seen in Figure 3, circles B and D are concentric.

Corollary 1.4 *Circle D will be defined according to Lemma 1 as*

$$(y-r)^2 + x^2 = \frac{1}{4}r^2 \tag{14}$$

Using the same method as for Corollary 1.2, the points of intersection can be calculated as follows:

$$\left(y - \frac{1}{2}r\right)^2 + x^2 = (y-r)^2 + x^2 \tag{15}$$

$$y^2 - yr + \frac{1}{4}r^2 = y^2 - 2yr + r^2 \tag{16}$$

$$yr = \frac{3}{4}r^2 \tag{17}$$

$$y = \frac{3}{4}r \tag{18}$$

And the x-coordinate is calculated as:

$$(y-r)^2 + x^2 = \frac{1}{4}r^2 \tag{19}$$

$$\left(\frac{3}{4}r - r\right)^2 + x^2 = \frac{1}{4}r^2 \tag{20}$$

$$\frac{1}{16}r^2 + x^2 = \frac{1}{4}r^2 \tag{21}$$

$$x^2 = \frac{3}{16}r^2 \tag{22}$$

$$x = \pm\frac{\sqrt{3}}{4}r \tag{23}$$

Corollary 1.5 *The circle C and Circle D will intersect each other at points* $\left(\frac{\sqrt{3}}{4}r, \frac{3}{4}r\right)$ *and* $\left(-\frac{\sqrt{3}}{4}r, \frac{3}{4}r\right)$.

At this point, the first two "layers" in this process has been made (Figure 3). Here, it will be further proven that the point of intersection of A and B, point of intersection of C and D, and the center of D (which is the upper circle of the inner layer) lie on a same line (Figure 5).

Lemma 2 *Point* γ *are said to be in a line between points* ρ *and* ϕ *if the coordinate for* γ *satisfies the equation of a line that connects points* ρ *and* ϕ.

Here suppose we find an equation of a line that connects the center of D to one of the points of intersection between A and B – here, the author will choose point $\left(\frac{\sqrt{3}}{2}r, \frac{1}{2}r\right)$ – but the same reasoning can be applied to other sets of points as well. The center of D, according to Corollary 1.4 is $(0, r)$ The slope of said line is found as:

$$m = \frac{y_1 - y_2}{x_1 - x_2} \tag{24}$$

$$m = \frac{\frac{1}{2}r - r}{\frac{\sqrt{3}}{2}r - 0} \tag{25}$$

$$m = \frac{-\frac{1}{2}r}{\frac{\sqrt{3}}{2}r} = -\frac{1}{\sqrt{3}} \tag{26}$$

The intercept of this line can be found using center of D as:

$$y = mx + (\text{y-intercept}) \tag{27}$$

$$r = -\frac{1}{\sqrt{3}} \cdot (0) + (\text{y-intercept}) \tag{28}$$

$$\text{y-intercept} = r \tag{29}$$

Corollary 2.1 *The equation of line that connects the center of D to a point of intersection between A and B is*

$$y = -\frac{1}{\sqrt{3}}x + r \tag{30}$$

To check that the point of intersection between C and D lies on this line, (x, y) is substituted with the coordinate for the point of intersection between C and D, which is $\left(\frac{\sqrt{3}}{4}r, \frac{3}{4}r\right)$. Therefore:

$$y \stackrel{?}{=} -\frac{1}{\sqrt{3}}x + r \tag{31}$$

$$\frac{3}{4}r \stackrel{?}{=} -\frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{4}r\right) + r \tag{32}$$

$$\frac{3}{4}r \stackrel{?}{=} -\frac{1}{4}r + r \tag{33}$$

$$\frac{3}{4}r = \frac{3}{4}r \tag{34}$$

Corollary 2.2 *The point of intersection between A and B, point of intersection between C and D, and the center of D lies on the same line.*

Lemma 3 *Due to the fact that the inner layers are similar figures to the outer layers, even if the process of adding "layers" are repeated more times, all points of intersections between the two circles of any layers will line up.*

Because of the conclusion achieved in Lemma 3, if the process is repeated such that the number of layers approaches ∞, the points between each layer, especially where it is closest to the center of the upper circles in each layer, will become closer and closer until said points approach the point where they make up an infinitesimal line segment. As such the theorem set forth is proven.

4 Conclusion

In conclusion, through this exploration of limits in geometry, it is proven that through the repeating steps set forth in this paper, an infinitesimal-length straight line can be created from infinite layers of two-circle, and that any numbers of intersection points between the two-circle in any number of layers done according to the process described herein are always in a straight line.

5 Acknowledgement

- Zachary M. Nooyen for engaging the author on this mathematical journey by asking "How was the first straight line created? [*sic*] Like if they don't have a straight thing to draw a straight line with, how did they do it," which got the author thinking if it is possible to create a straight line with infinitely many circles.